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Materials with a desired refraction coefficient can be created by embedding small particles into a given material

A G Ramm

Department of Mathematics, Kansas State University, Manhattan, KS 66506-2602, USA

Abstract

A method is given for creating material with a desired refraction coefficient. The method consists of embedding into a material with known refraction coefficient many small particles of size a, $ka \ll 1$, where $k > 0$ is the wave number. The number of particles per unit volume around any point is prescribed, the distance between neighboring particles is $O(a^{\frac{2-\kappa}{3}})$ as $a \to 0$, $0 < \kappa < 1$ is a fixed parameter. The total number of the embedded particle is $O(a^{\kappa-2})$. The physical properties of the particles are described by the boundary impedance ζ_m of the $m-th$ particle, $\zeta_m = O(a^{-\kappa})$ as $a \to 0$. The refraction coefficient is the coefficient $n^2(x)$ in the wave equation $[\nabla^2 + k^2 n^2(x)]u = 0$. Technological problems, which should be dealt with before the method can be implemented practically, are formulated and discussed. The desired refraction coefficient can be complex-valued. This means that the enrgy absorption in the new material can also be designed as one wishes.

Key words: metamaterials, refraction coefficient, wave scattering, small particles.

1. Introduction

The problem we are concerned with is the following:

How does one create in a given bounded domain $D \subset \mathbb{R}^3$ a material with a desired refraction coefficient $n^2(x)$? What are the technological problems to be solved in order to implement practically our recipe for creating materials with a desired refraction coefficient?

Initially the domain D is assumed to be filled in with a material with a known refraction coefficient $n_0^2(x)$. It is assumed that $\operatorname{Im} n_0^2(x) \geq 0$ and $n_0^2(x) = 1$ in $D' := \mathbb{R}^3 \setminus D$. The wave equation in this material is:

$$L_0 u_0 := [\nabla^2 + k^2 n_0^2(x)]u_0 = 0 \quad \text{in } \mathbb{R}^3, \quad k = const > 0, \tag{1}$$

$$u_0 = e^{ik\alpha\cdot x} + v_0, \quad \alpha \in S^2, \tag{2}$$

$$v_0 = A_0(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \quad \beta := \frac{x}{r}. \tag{3}$$

The function v_0 is the scattered field, $A_0(\beta, \alpha, k)$ is the scattering amplitude, $u(x, \alpha, k)$ is the scattering solution, S^2 is the unit sphere in $\mathbb{R}^3$.

We embed M small particles D_m, $S_m := \partial D_m$, $1 \leq m \leq M$, into D, so that in any subdomain $\Delta \subset D$ there are

$$\mathcal{N}(\Delta) = \frac{1}{a^{2-\kappa}} \int_\Delta N(x)dx[1 + o(1)], \qquad a \to 0 \tag{4}$$

Email address: ramm@math.ksu.edu (A G Ramm)

small particles. Here $N(x) \geq 0$ is a continuous (or piecewise-continuous) function which we can choose as we wish, $0 < \kappa < 1$ is a parameter which is our disposal. For simplicity we assume that particles D_m are balls centered at the points x_m and of radius a independent of m. The embedded particles are small, $ka \ll 1$, where $k = \frac{2\pi}{\lambda}$, λ is the wavelength in the material in D, $k > 0$ is the wavenumber.

Our theory can be generalized to the case of small particles of arbitrary shapes (Ramm (2005)). Some results on numerical modeling of our method are described in papers (Andriychuk and Ramm (2010a), Andriychuk and Ramm (2010b), and Indratno and Ramm (2010)).

The distance d between neighboring particles is assumed to be

$$d = O(a^{\frac{2-\kappa}{3}}) \quad \text{as} \quad a \to 0. \tag{5}$$

This assumption is dictated by the assumption (4). Indeed, the number of particles on a unit length linear segment is equal to $O(\frac{1}{d})$, if the distance between neighboring particles is $O(d)$. Therefore, the number of particles in a unit cube is $O(\frac{1}{d^3})$, and this number is equal to $O(\frac{1}{a^{2-\kappa}})$, according to formula (4). Consequently, d has to be given by formula (5).

The properties of a particle are described by the boundary impedance

$$\zeta_m = \frac{h(x_m)}{a^\kappa}, \tag{6}$$

where $h(x)$ is a continuous function on D, Im $h(x) \leq 0$. The function $h(x)$, as $N(x)$, we can choose as we wish.

The scattering solution $u(x, \alpha, k)$ in the presence of the embedded particles solves the problem:

$$L_0 u = 0 \quad \text{in } \mathbb{R}^3 \setminus \cup_{m=1}^{M} D_m, \tag{7}$$

$$u_N = \zeta_m u \quad \text{on } S_m, \quad 1 \leq m \leq M, \tag{8}$$

$$u = u_0(x, \alpha, k) + v, \tag{9}$$

$$v = A_1(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad |x| = r \to \infty, \quad \beta := \frac{x}{r}. \tag{10}$$

Let us now describe our results. We have proved that problem (7)-(10) has a unique solution $u(x, \alpha, k) := u_M(x, \alpha, k)$, (see paper (Ramm (2007a))). We have proved in paper (Ramm (2008b)) that given an arbitrary bounded function $n^2(x)$ such that $n^2(x) = 1$ in D', $n^2(x)$ is continuous or piecewise-continuous in D (with the set of discontinuities of Lebesgue measure zero in $\mathbb{R}^3$), one can choose $N(x)$ and $h(x)$ so that the limit

$$\psi := \psi(x, \alpha, k) = \lim_{M \to \infty} u_M(x, \alpha, k) \tag{11}$$

exists and satisfies the equation

$$[\nabla^2 + k^2 n^2(x)]\psi = 0 \quad \text{in } \mathbb{R}^3, \tag{12}$$

$$\psi = u_0(x, \alpha, k) + w(x, \alpha, k), \tag{13}$$

$$\psi = e^{ik\alpha\cdot x} + A(\beta, \alpha, k)\frac{e^{ikr}}{r} + o\left(\frac{1}{r}\right), \quad r = |x| \to \infty, \quad \beta := \frac{x}{r}. \tag{14}$$

Therefore the medium with embedded particles in the limit $M \to \infty$, or, which is the same by (4), in the limit $a \to 0$, has a desired refraction coefficient $n^2(x)$. The refraction coefficient $n^2(x)$ can be a complex-valued function, so that the absorption of energy in the limiting material can be designed as we wish. The imaginary part of the refraction coefficient comes from the imaginary part of the function $h(x)$, defining the boundary impedances, see in Section 2 **Steps** 1 and 2 of the recipe for creating material with a desired refraction coefficient. Our method allows one to create a material with a refraction coefficient which is a tensor, but we do not go into detail. In recent papers (Ramm (2010b) and Ramm (2010c)) one finds a similar theory for electromagnetic (EM) wave scattering by many small particles, and a recipe for creating materials with a desired refraction coefficient for EM wave scattering.

In practice, one has to stop at some small but finite $a > 0$ and at the total number $M = M(a)$ of the embedded particles. The corresponding refraction coefficient $n^2_M(x)$ will approximate the desired refraction coefficient $n^2(x)$ with an error that tends to zero as $a \to 0$.

In Section 2 we formulate *the recipe* for choosing $N(x)$ and $h(x)$ which guarantees the existence of the limit (11), that is, the existence of the limiting function $\psi(x, \alpha, k)$. This function solves problem (12)-(14).

Our theory resembles a homogenization theory (see, e.g., papers (Milton (2001), Marchenko and Khruslov (2006) and Kozlov, et al. (1994))). However, there are essential differences compared with the usual theories in these references: a) we do not assume that the small particles are embedded periodically, b) we do not assume that the operators involved are selfadjoint and have discrete spectrum, c) the estimates for the scattering solution to problem (7)-(10) differ from the usual estimates in the homogenization theory.

The aims of this paper are:

1) To make clear for a wide audience of engineers and physicists our recipe for creating material with any desired refraction coefficient,

and

2) To formulate two technological problems which must be solved in order that our theory can be immediately implemented experimentally.

Theoretical justification of our results are given in papers (Ramm (2007a), Ramm (2007b), Ramm (2008a) and Ramm (2008b), see also Ramm (2007c), Ramm (2007d) and Ramm (2005) and the papers by the author mentioned in References. In book (Landau, et al. (1984)), one finds basic physical theory of electromagnetic wave propagation in an inhomogeneous medium.

2. The recipe for creating material with a desired refraction coefficient

The problem we are interested in is the following:

One is given $n^2_0(x)$ and wants to create a refraction coefficient $n^2(x)$.

Here is our recipe for doing this.

Step 1. Calculate the function

$$p(x) := k^2[n^2_0(x) - n^2(x)] := p_1(x) + ip_2(x), \tag{15}$$

where $p_1(x) = \text{Re } p(x)$, $p_2 = \text{Im } p(x)$.

Step 2. Find two functions $N(x) \geq 0$ and $h(x) = h_1(x) + ih_2(x)$ from the relation

$$4\pi h(x)N(x) = p(x). \tag{16}$$

This can be done by infinitely many ways. For example, one may fix $N(x) > 0$ and define

$$h_1 = \frac{p_1(x)}{4\pi N(x)}, \quad h_2 = \frac{p_2(x)}{4\pi N(x)}. \tag{17}$$

If one wishes to deal only with passive materials, then one requires $\text{Im } n^2(x) \geq 0$, $\text{Im } h(x) \leq 0$, and, if $\text{Im } n^2_0(x) \leq \text{Im } n^2(x)$,then $\text{Im } p(x) \leq 0$.

Step 3. Partition the domain D into a union of small cubes Δ_p, $1 \leq p \leq P$, without common interior points, $D = \cup_{p=1}^{P} \Delta_p$, the center of Δ_p is denoted by y_p, the side of Δ_p is of the order $O(a^{\frac{2-\kappa}{6}})$. In each cube Δ_p embed $\mathcal{N}(\Delta_p)$ small particles, where $\mathcal{N}(\Delta_p)$ is defined in (4). The distance d between neighboring particles should be $d = O(a^{\frac{2-\kappa}{3}})$. The given order of the smallness of d as $a \to 0$ is important, but the distance need not be exactly the same. The boundary impedance of each of the small particle embedded in Δ_p make equal to $\frac{h(y_p)}{a^\kappa}$, where $h(x) = h_1(x) + ih_2(x)$ is the function found in Step 2 of the recipe.

Theorem 2.1. *After the completion of Step 3, the material, obtained from the original one with the refraction coefficient $n_0^2(x)$, will have the refraction coefficient $n_M^2(x)$, and $\lim_{M\to\infty} n_M^2(x) = n^2(x)$.*

Proof of this theorem one finds in papers (Ramm (2007a) and Ramm (2008b)). Numerical modeling and results concerning wave scattering by many small particles can be found in papers Andriychuk and Ramm (2010a), Andriychuk and Ramm (2010b), and Indratno and Ramm (2010). Generalizations to the case of electromagnetic (EM) wave scattering by many small particles is given in recent papers Ramm (2010b) and Ramm (2010c).

Example. Let us illustrate our method by a simple example. Suppose that one wants to create a material with the refraction coefficient $n^2(x) = 1 + x_3$ in the region D, described by the inequalities $0 \le x_3 \le 1, 0 \le x_1, x_1 \le 1$, $x = (x_1, x_2, x_3)$. Assume that $n_0^2(x) = 1$ in D, and $k^2 = 1$. Choose $N(x) = 1$ in D. Calculate $p(x) = k^2[n_0^2(x) - n^2(x)] = -x_3$, where we took into account that $k^2 = 1$. Thus, $p_2(x) = 0$, $p_1(x) = -x_3$. Use equation (17) to calculate $h(x)$ and get $h(x) = -\frac{x_3}{4\pi}$. This completes **Steps** 1 and 2 of the recipe. By embedding in the cube D small particles according to **Step** 3 of the recipe, one gets a material with the desired refraction coefficient $n^2(x) = 1 + x_3$ in the region D.

3. A discussion of the recipe

Step 1 of the recipe is trivial. Step 2 is also trivial. One may choose $N(x) > 0$ to satisfy some practical requirements. For example, if one chooses $N(x)$ small, then the total number of particles will be smaller. Practically one cannot take the limit $M \to \infty$, i.e., in the limit $a \to 0$, and one stops at some finite value of M, or of $a > 0$. If one stops at a sufficiently small a, then Theorem 2.1 implies that the resulting medium will have the refraction coefficient $n_a^2(x)$, such that the relative error $\frac{|n_M^2(x)-n^2(x)|}{n^2(x)}$ is arbitrarily small. Experimental implementation of our theory should show at what size of a one should stop in practice.

The *two technological problems*, that have to be solved in order that our recipe can be implemented experimentally, are:

1) How does one embed a small particle at a given point into the given material in D?

2) How does one prepare a small particle, a ball of radius a centered at a point x_m, with the prescribed boundary impedance $\zeta_m = \frac{h(x_m)}{a^\kappa}$?

Here $h(x)$ is the function, found at Step 2 of the recipe.

Possibly, the first technological problem can be solved by the *stereolitography* process. This process allows one to create at a precisely desirable location in some materials a small particle of nano-size.

One should be able to solve the second technological problem because its limiting cases $\zeta = 0$ (acoustically hard particles, particles from insulating material) and $\zeta_m = \infty$ (acoustically soft particles, perfectly conducting particles) are easy to solve in practice, so the intermediate values of the boundary impedance should be also possible to prepare.

A similar theory has been developed in paper Ramm (2008a) for electromagnetic (EM) wave scattering by many small dielectric and conducting particles embedded in an inhomogeneous medium. In recent papers (Ramm (2010b) and Ramm (2010c)) one finds a theory for EM wave scattering by many small impedance particles, and a recipe for creating materials with a desired refraction coefficient and a desired magnetic permeability for EM wave scattering. A brief summary of the results of this theory is given at the end of this paper.

4. Electromagnetic waves

Assume now that the governing equations are the Maxwell equations

$$\nabla \times E = i\omega\mu H, \quad \nabla \times H = -i\omega\epsilon'(x)E \qquad in \quad \mathbb{R}^3, \tag{18}$$

$\mu = const$, $\epsilon'(x) = \epsilon = const$ in D', $\omega > 0$ is frequency, $\epsilon'(x) = \epsilon(x) + i\frac{\sigma(x)}{\omega}$, $\sigma(x) \ge 0$ is the conductivity, $\sigma(x) = 0$ in D'. We assume that $\epsilon'(x) \in C^2(\mathbb{R}^3)$, $\epsilon'(x) \ne 0$, is a twice continuously differentiable function. Let $k = \frac{\omega}{c}$, $c = \omega\sqrt{\epsilon\mu}$ is the wave velocity in D'. The incident plane wave is $\mathcal{E}e^{ik\alpha\cdot x}$, $\alpha \in S^2$, $\alpha \cdot \mathcal{E} = 0$, $\mathcal{E}$ is a constant vector.

Under the above assumptions the electrical field $E(x)$ is the unique solution to the equation (see paper (Ramm (2008a))):

$$E_0(x) = \mathcal{E}e^{ik\alpha\cdot x} + \int_D g(x,y)p(y)E_0(y)dy + \nabla_x \int_D g(x,y)q(y)\cdot E_0(y)dy, \qquad g(x,y) := \frac{e^{ik|x-y|}}{4\pi|x-y|}, \tag{19}$$

where

$$p(x) := K^2(x) - k^2, \quad K^2(x) := \omega^2\epsilon'(x)\mu; \quad q(x) := \frac{\nabla K^2(x)}{K^2(x)}. \tag{20}$$

If M small particles D_m, $1 \leq m \leq M$, are embedded in D, then the basic equation (19) becomes

$$E_M(x) = E_0(x) + \sum_{m=1}^{M}\int_{D_m} g(x,y)p(y)E_M(y)dy + \sum_{m=1}^{M}\nabla_x\int_{D_m} g(x,y)q(y)\cdot E_M(y)dy. \tag{21}$$

It is proved in paper (Ramm (2008a)) that if the size a of small particles tends to zero, if the number of these particles in any open subset Δ of D is

$$\mathcal{N}(\Delta) = \frac{1}{a^{3-\kappa}}\int_\Delta N(x)dx[1+o(1)], \quad a\to 0, \tag{22}$$

and if the distance d between neighboring particles is $d = O(a^{\frac{3-\kappa}{3}})$, then there exists the limit $\lim_{M\to 0} E_M(x) = E_e(x)$.

The limiting field $E_e(x)$, i.e., the effective field in the medium, solves the equation:

$$E_e(x) = E_0(x) + \int_D g(x,y)C(y)E_e(y)dy, \quad C(y) = N(y)c(y), \tag{23}$$

where

$$c(y) = \lim_{a\to 0}\frac{1}{a^{3-\kappa}}\int_{|y-x|\leq a} p(x)dx. \tag{24}$$

If, e.g., the small particle D_m is a ball of radius a centered at a point y, and

$$p(x) = \begin{cases} \frac{\gamma(y)}{4\pi a^\kappa}\left(1-\frac{|x|}{a}\right)^2, & |x|\leq a; \\ 0, & |x| > a, \end{cases}$$

in the coordinate system with the origin at the point y, and $\gamma(y)$ is a number we can choose as we wish, then $c(y)$ in (24) can be easily calculated: $c(y) = \gamma(y)/30$. Equation (23) implies:

$$[\nabla^2 + \mathcal{K}^2(x)]E_e = 0 \quad \text{in } \mathbb{R}^3, \quad \mathcal{K}^2(x) := K^2 + C(x), \tag{25}$$

where $C(x)$ is defined in (23). This equation can be rewritten as

$$\nabla\times\nabla\times E_e = \mathcal{K}^2(x)E_e + \nabla\nabla\cdot E_e. \tag{26}$$

The term $\nabla\nabla\cdot E_e$ plays the role of the current $i\omega\mu J$.

This term can also be interpreted as the term due to a non-local susceptibility χ: if

$$D_e(x) = \tilde{\epsilon}(x)E_e - i\omega\int_D \chi(x,y)E_e(y)dy,$$

then the Maxwell's equations

$$\nabla\times E_e = i\omega\mu H_e, \quad \nabla\times H_e = -i\omega\tilde{\epsilon(x)}E_e - i\omega\int_D \chi(x,y)E_e(y)dy$$

imply

$$\nabla \times \nabla \times E_e = \omega^2 \tilde{\epsilon}(x) \mu E_e(x) + \omega^2 \mu \int_D \chi(x, y) E_e(y) dy.$$

This equation is of the form (26) if $\tilde{\epsilon}(x) = \frac{\mathcal{K}^2(x)}{\omega^2 \mu}$, and

$$\chi(x, y) = (\omega^2 \mu)^{-1} \nabla_x (\delta(x - y) \nabla_y),$$

where $\delta(x - y)$ is the delta-function.

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